Robust nonlinear control of shunt active filters for harmonic current compensation

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Abstract

An advanced control strategy of shunt active filters (SAF) aiming to compensate for harmonic current in the electric supply grid is proposed. The SAF considered here is suitable for three-phase three-wire current harmonic compensation and is based on AC/DC three-phase boost converter topology. Robust control of the active-reactive current/power delivered by the SAF is designed exploiting the internal model principle. The stabilization of the DC-link voltage dynamics is addressed along with the fulfillment of the harmonics compensation objective. The two-time scale behavior of the SAF is exploited to apply the averaging theory in the control design. Experiments are provided to show the effectiveness of the proposed solution.

Keywords: Shunt active filter control; Robust control; Output regulation; Singular perturbation; Averaging theory

1. Introduction

Nonlinear loads connected to AC electric mains generate undesirable harmonics in the current dynamics which are usually responsible of additional power losses and the risk of equipment damage or malfunctioning. In this respect a remarkable research attempt has been devoted, both from a technological and theoretical point of view, to identify countermeasures reducing the level of harmonic current drawn from the electric mains. Traditionally, current harmonics have been compensated with passive filters which, as all passive devices, have several intrinsic limitations rendering their use ineffective in realistic situations characterized by different and not a priori fixable operative modes. In the last decades, the fast development of power electronics components and control processors has led to a growing interest in the so-called active power filters (APF) (see Singh & Al-Haddad, 1999). These devices are potentially able to properly work and to guarantee performances in a wide range of operating conditions overtaking in this way intrinsic limitations of passive devices.

A particular kind of APF are the shunt active filters (SAF) whose purpose is to inject into the mains a proper current/power in order to compensate partially or totally for the harmonic current generated by nonlinear loads (see Mattavelli, 2001; Mendalek & Al-Haddad, 2000). The SAF considered in this paper is based on three-phase three-wire AC/DC boost converter topology and it is connected in parallel with distorting loads as it is shown in Fig. 1. This kind of devices is suitable for three-phase, three-wire harmonic compensation, i.e. it is suitable for AC three-phase line grid and neglect current in the neutral. Different approaches dealing with this kind of SAF have been presented in literature (see Singh & Al-Haddad, 1999). In this respect, the main distinguishing marks are the filter current control algorithm and the load current analysis methods adopted to determine the filter current objective.

In Chandra, Singh, and Al-Haddad (2000) the high performances of hysteresis current controllers (Kazmierkowski & Malesani, 1998) are exploited. In Jeong and Woo (1997) predictive current control is adopted. For the generation of filter
current reference, the instantaneous power theory (Akagi & Nabae, 1986), the time-domain correlation technique (Van Harmelen & Enslin, 1993), the notch filter theory to remove the fundamental current component (Rastogi, Mohan, & Edris, 1995), and some other methods have been used to detect the load current harmonics. In Bhattacharya, Veltman, Divan, and Lorentz (1995) an observer of the inductor fluxes is used for both reference generation and SAF current/voltage stabilization are “decoupled” using Power/current control, used for load harmonics compensation, a functional integral controller with a suitably designed deadzone.

The objectives are discussed and “adapted” to cope with the in-ternal dynamics is unstable when perfect harmonic compensation changes energy with the line and the load to compensate for the parasitic resistive power losses, the voltage internal dynamics is unstable when perfect harmonic compensation is achieved.

In this paper the overall SAF control problem is addressed. The scheme of the SAF considered in this article and its connection to an AC three-phase line grid where nonlinear loads are present are shown in Fig. 1. The SAF is a three-phase three-wire AC/DC boost converter, where the capacitor is the main energy storage element and the inductors are used to control the current filters by means of the converter voltages. The switches of the three-legs-bridge (commonly indicated as “inverter”) are usually realized by IGBTs and free-wheeling diodes. In this figure \( v_s = (v_{sa}, v_{sb}, v_{sc})^T \) are the sinusoidal mains voltages, \( i_s = (i_{sa}, i_{sb}, i_{sc})^T \) are the mains currents, \( \ell = (\ell_a, \ell_b, \ell_c)^T \) are the filter currents, \( v \) is the capacitor voltage, \( \ell_c = (\ell_{ca}, \ell_{cb}, \ell_{cc})^T \) are the nonlinear-load currents.\(^1\) The inverter leg switch commands are denoted by \( u_t = (u_s, u_y, u_z)^T \), where \( u_s, u_y, u_z \in \{0, 1\} \); \( 0 = \text{OFF}, 1 = \text{ON} \). Since a PWM strategy is assumed to control the inverter, the above-mentioned commands will be considered such that \( u_s, u_y, u_z \in [0, 1] \). Furthermore, in accordance with standard SAF characteristics, the filter and load currents \( i \) and \( \ell_c \), the line voltages \( v_s \), and the DC-link voltage \( v \), can be assumed all measurable and available for feedback.

The physical parameters mainly characterizing the SAF are the values of the inductance \( L \), of the DC-link capacitor \( C \) and of the parasitic resistance \( R \) affecting the inductors and the switches. According to the usual characteristic of power components, the value of \( C \) can be assumed known with very good accuracy, while the value of \( L \) and, above all, \( R \) are usually rather uncertain. In the forthcoming discussion we will assume the value of \( R \) and \( L \) unknown but constant and ranging in known bounded sets.

The mathematical model of the three-wire SAF shown in Fig. 1 can be easily derived using a “power representation” obtained by changing the current variables \( i \) into two-phase real-imaginary power variables defined as \( x = (x_p, x_q) \)

\[
p_{T_{abc}}(t) = \frac{2V_s}{3} \begin{pmatrix} \cos \omega_s t & \sin \omega_s t & (1 & -1/2 & -1/2 \\ -\sin \omega_s t & \cos \omega_s t & 0 & \sqrt{3}/2 & -\sqrt{3}/2) \end{pmatrix} \]

in which \( V_s > 0 \) and \( \omega_s = 2\pi f_s \) are known constants (\( f_s = 50 \) or \( 60 \) Hz depending on the country). In this coordinate setting, by taking advantage of the hypothesis of three-phase balanced sinusoidal line, the model of the SAF filter shown in Fig. 1 is described by (see Marconi et al., 2004, Tilli et al., 2002)

\[
\dot{x} = M(R, L)x - \frac{v}{L} p_{T_{abc}} u + d_0,
\]

\[
\dot{v} = \frac{e^T}{2} p_{T_{abc}}^T x,
\]

\( \text{Section 3. Experimental results showing the effectiveness of the proposed solution are shown and discussed in Section 4.} \)

\( \text{Fig. 1. Shunt active filter scheme.} \)

\( \text{2. The framework} \)

\( \text{2.1. Shunt active filter and nonlinear load models} \)

\( \text{The paper is organized as follows. In the next section, the general framework is presented while the controller design is reported in Section 3. Experimental results showing the effectiveness of the proposed solution are shown and discussed in Section 4.} \)
where \( \bar{u} = u_t - \frac{1}{3}(1 1 1)^T(u_x + u_y + u_z) \),
\[
d_0 = \begin{pmatrix} E_{sp}/L & 0 \\ 0 & M(R, L) \end{pmatrix}, \quad M(R, L) = \begin{pmatrix} -R/L & \omega_s \\ -\omega_s & -R/L \end{pmatrix}
\]
\( \epsilon = 3/(C E_{sp}) \) and \( E_{sp} = V_y^2 \). On this model the controller will be designed. To this regard it is worth noting that, as the filter currents \( i \), the line voltages \( v \), and the DC link voltage \( v \) are all measurable, it turns out that the full state \((x, v)\) of (1) is perfectly known and available for feedback. Furthermore note that the actual control action \( u_t = (u_x, u_y, u_z)^T \) can be effectively determined from \( \bar{u} \) (Kazmierkowski & Malesani, 1998).

As far as the load description is concerned, we will assume, following (Akagi et al., 1984), that the load currents are approximated as periodic signals with period \( 1/f_s \) given by the sum of a finite number \( N \) of harmonics at frequencies multiple of \( f_s \). More in detail we will assume that the equivalent two-phase real-imaginary power representation of the load current, defined as \( x_l = (x_{lp} x_{lq})^T \equiv pT_{abc}^{*}t \), has the form
\[
x_l(t) = x_{l0} + \sum_{n=1}^{N} x_{jn} \cos(n\omega_s t + \psi_{jn}), \quad j = p, q
\]
in which \( x_{l0}, X_{lp0}, X_{lq0}, X_{lpn}, X_{lqn}, \psi_{pn} \) and \( \psi_{qn} \) are constants.

Remark. It turns out that the constant term \( x_{lp0} \) is the only component contributing to deliver energy to the load while the oscillatory part \( x_{lp}(t) - x_{lp0} \) of the real component, which has a null balance over each line period, and the imaginary power component \( x_{lq}(t) \) do not contribute to power flow. In particular, the imaginary power component \( x_{lq}(t) \) represents a measure of “misalignment” between line voltages and load currents (see (Mohan, Undeland, & Robbins, 1989) for more details). In this respect the components \( x_{lp}(t) - x_{lp0} \) and \( x_{lq}(t) \) represents undesired load components which should be compensated by properly designing the filter currents (see Section 2.2).

Finally note that, since the load currents \( i \) and the line voltages \( v \) are measurable, the variables \( x_l = (x_{lp}, x_{lq})^T \) are known and available for feedback.

2.2. Problem formulation

Roughly speaking, the control problem amounts to fulfill the following two objectives:

(a) to compensate the load harmonic currents by injecting suitable currents into the line grid. This control objective qualifies as a tracking problem in which the reference for the filter currents are the undesired load harmonics opposite in phase;

(b) to guarantee that the value of the DC link voltage \( v(t) \) remains confined in a region \([V_m, V_M]\), with \( V_M > V_m > 0 \), to avoid the complete discharge and an overcharge of the capacitor.

The two control objectives are strongly interrelated: on the hand the capability of maintaining the DC link voltage value in the desired range relies upon the power absorbed/delivered to the mains which, in turn, is strictly related to the load harmonics to be compensated for. On the other hand, the ability of tracking currents references relies upon the value of the DC link voltage which is the main power source of SAF. Goal of the next part of the section is to precisely investigate the properties and the inherent constraints of the system in order to formulate a precise and feasible control problem.

By bearing in mind the remark at the end of the previous section, it would be natural to re-formulate the control objective (a) in ideal way as an asymptotic state tracking problem for system (1) of the ideal reference
\[
x^*(t) = \begin{pmatrix} x_{p}^*(t) \\ x_{q}^*(t) \end{pmatrix} = \begin{pmatrix} X_{lp0} - x_{lp}(t) \\ -x_{lq}(t) \end{pmatrix}^T.
\]

This, indeed, would guarantee that the mains currents are definitely constituted by a pure sinusoid perfectly aligned with the line voltages. This ideal control objective, though, can be easily discovered to be in contrast with the control objective (b). To motivate this claim, consider the tracking dynamics of system (1) with respect to the reference \( x^* \), namely the steady state system dynamics compatible with the constraint \( x(t) \equiv x^*(t) \), which, as a simple computation shows, are given by
\[
\frac{dx^2(t)}{dt} = \epsilon L(d_0 + M(R, L)x^*(t) - \dot{x}^*(t))^T x^*(t)
\]
\[
: = e\Psi(x^*(t)).
\]

System (3) is a simple integrator driven by the periodic signal \( e\Psi(x^*(t)) \) with period \( 1/f_s \) given by the sum of the periodic signals \( e\Psi(L(d_0 - \dot{x}^*(t))^T x^*(t) \) with zero mean value, and \( e\Psi(M(R, L)x^*(t))^T x^*(t) \) whose mean value is negative as long as \( x^*(t) \) or the parasitic resistance \( R \) are not identically zero. By this it turns out that, no matter which initial condition is taken in the set \([V_m, V_M]\), there exists a finite time \( t \) such that \( v(t) < V_m \). The physical interpretation of this phenomenon is that the power losses related to parasitic resistance \( R \) (and, other possible non-idealities neglected in the model (1)), generate a continuous discharge of the SAF DC-link capacitor when the asymptotic power tracking objective is pursued which leads to a loss of controllability of the system. To avoid this undesired phenomenon the ideal tracking objective must be revised by taking into account, in the power reference signal, an additional power term which should be drawn from the line grid by the SAF to compensate for its power losses. Following this motivation, and by bearing in mind the Remark at the end of Section 2.1, the power reference signal (2) is modified as
\[
x_{p0}^* (t) = x^*(t) + (\phi_0t)^T
\]
in which \( \phi_0 \) is a solution of the second order polynomial
\[
R\phi_0 - E_{sp}\phi_0 + Rf_s \int_{0}^{1/f_s} (x_{p}^2(\tau) + x_{q}^2(\tau)) d\tau = 0.
\]

Eq. (5) represents a power balancing condition (see also Section 2.3) which guarantees that the tracking dynamics of system (1) with respect to the modified reference \( x_{p0}^* \) is
given by
\[
\frac{dv^2(t)}{dt} = e\mathcal{P}(x_{\phi_0}^*(t))
\] (6)
in which the drift right-hand side term \(e\mathcal{P}(x_{\phi_0}^*(t))\) is a periodic signal with period \(1/f_s\) with zero mean value. From this, simple computations can be used to show that, if \(e\) is sufficiently small (namely \(C\) sufficiently large) compared with the integral of \(\mathcal{P}(x_{\phi_0}^*(t))\) over a period of length \(1/f_s\), then there exists an initial condition \(v(0) \in [V_m, V_M]\) yielding \(v(t) \in [V_m, V_M]\) for all \(t \geq 0\). In others words the tracking of the modified reference \(x_{\phi_0}^*(t)\) is potentially achievable by maintaining the DC link voltage in the desired range \([V_m, V_M]\). The value \(\phi_0\) is precisely the extra power to be drawn by the SAF to compensate for power losses.

Motivated by these arguments we precisely formulate the control problem which will be faced throughout the paper as the one of designing the control input \(\bar{u}\) in such a way the following two objectives are fulfilled:

(O1) given the reference signal \(x_{\phi_0}^*\), defined in (2), (4), (5), asymptotic state tracking must be achieved, namely
\[
\lim_{t \to \infty} (x(t) - x_{\phi_0}^*(t)) = 0;
\]

(O2) given the safe voltage interval \([V_m, V_M]\), with \(V_M > V_m > 0\) and assuming \(v(t_0) \in [V_m, V_M]\), it must be \(v(t) \in [V_m, V_M]\), \(\forall t > t_0\).

We conclude this section with a few remarks aiming to better frame the control problem.

**Remark.** As far as (O1) is concerned, it is worth noting that neither \(x_{\phi_0}^*\) (depending on \(R\) through (5)) and the tracking error \(x(t) - x_{\phi_0}^*(t)\) (which is not measurable as such) are known. In this respect, the controller is required to posses the ability of dynamically reconstructing the reference \(x_{\phi_0}^*(t)\) by elaborating the DC-link voltage value estimating, in this way, the power losses due to the parasitic resistance.

**Remark.** In the forthcoming design the constraints on the actual control action \(u_t\) imposed by the PWM strategy will be not explicitly taken into account. In doing this, as usual in the control of power electronics devices, we will assume that the PWM constraint \(u_t \in [0, 1]\) can be met by suitably choosing the lower DC-link voltage bound \(V_m\), depending on the maximum nonlinear load to be compensated (see Ronchi & Tili, 2002).

**Remark.** Finally, note that the requirement in (O2) asking that \(v(t_0) \in [V_m, V_M]\) is not restrictive at all. In fact, following the theory of AC/DC boost converters (Mohan et al., 1989), it turns out that the natural un-controlled response of the system leads the DC-link voltage to a level which is twice the line-to-line peak voltage (due to \(L-C\) resonance and free-wheeling diodes of the inverter). This value, if a good system design has been performed, is expected to be greater than \(V_m\). Thus, after an initial uncontrolled transient, the controller can be switched on having \(v(t_0) \in [V_m, V_M]\).

2.3. A few considerations about the values of the physical parameters

In this part we present a few considerations about the values of the physical parameters which can be helpful as guidelines for system dimensioning and to motivate some of the conditions assumed in the forthcoming stability analysis.

We begin with some remarks about the value of \(V_m\) and \(V_M\) entering in (O2). With an eye to model (1) note that, as the control action \(\bar{u}\) is saturated, the lower bound \(V_m\) must be greater than a certain value depending on the saturation level, on the SAF parameters (basically the greatest possible values of \(R\) and \(L\)), on the line voltage and on the admissible set of references \(x^*\) (namely on the features of the nonlinear load). Differently, the upper bound \(V_M\) only depends on the maximum ratings of the adopted capacitor. More precise guidelines on the choice of \(V_m\) and \(V_M\) can be found in Ronchi and Tili (2002).

A crucial role in the operating mode of the SAF and in the forthcoming stability analysis is played by the value of the capacitor \(C\) (or, equivalently, \(\epsilon\)). In particular, as the capacitor is the main energy storage element of the filter, it is expected that the value of \(\epsilon\) is large (small). By bearing in mind the discussion above (and in particular (6)), a crucial requirement in this sense is that the value of \(\epsilon\) is sufficiently small so that the steady state oscillatory behavior of the DC link voltage can be always confined in the safe region \([V_m, V_M]\). In addition, all the stability analysis presented in the paper relies upon a *time-scale separation* condition asking that the current dynamics are sufficiently faster than the voltage dynamics. These conditions, more precisely set in the next section, are additional constraints asking for large values of \(C\).

Finally, we make a few considerations about the solutions \(\phi_0\) of the second order equation (5). The latter turns out to have two real positive solutions if and only if the following relation holds:
\[
E_{sp}^2 \geq 4R^2L \int_0^{1/f_s} \left((x_p^2(\tau) + x_q^2(\tau)) \right) d\tau.
\] (7)

It is interesting to note that (7) has a very reasonable physical meaning setting an “upper bound” on the admissible un-desired load components which can be compensated by SAF and on the parasitic resistance \(R\) compared with the main voltage. In this respect note that, as typically \(E_{sp}/R \gg 1\), this condition is not limitative at all as always fulfilled in classical loads scenarios. Moreover, Eq. (5), under condition (7) and since \(E_{sp}/R \gg 1\), has a first solution
\[
\phi_0 \approx \frac{R}{E_{sp}} \int_0^{1/f_s} \left(x_p^2(\tau) + x_q^2(\tau) \right) d\tau \approx 0.
\] (8)
and a second solution \(\phi_0 \approx E_{sp}/R \gg 1\). From a physical viewpoint the first solution, minimizing the power drawn from the line grid to compensate losses, is the most meaningful and is the one which will be considered throughout the paper.

3. The robust controller design

We begin by considering the preliminary control law (always well-defined since we will take care that \(v(t) \geq V_m > 0\) for all
\( t \geq 0 \) according to objective O2

\[
\tilde{u} = \frac{1}{\nu} P T^{-1}_{abc} u
\]

in which \( P T^{-1}_{abc} \) is the right inverse of \( P T_{abc} \) and \( u \in \mathbb{R}^2 \) is the new control input, which transforms system (1) into

\[
\dot{x} = M(R, L)x - \frac{1}{L} u + d_0, \quad \frac{d u^2}{d t} = \nu u^T x.
\]  

(9)

Consider now a “voltage error variable” \( \tilde{z} \) defined as

\[
\tilde{z} := v^2 - \tilde{v}^2 \quad \text{with} \quad \tilde{v}^2 = \frac{V_M^2 + V_m^2}{2}
\]

and note that the requirement O2 of having \( v(t) \in [V_m, V_M] \) for all \( t \geq t_0 \) is equivalent to the requirement of having \( \tilde{z}(t) \in [-\ell^*, \ell^*] \) for all \( t \geq t_0 \) where \( \ell^* = (V_M^2 - V_m^2)/2 \).

With this notation in mind we fix a known reference signal for the power delivered by the filter as

\[
x_{\eta} = x^* - N_1 q(z) + N_1 \eta,
\]

(11)

where \( N_1 = (10)^T, q(s) : \mathbb{R} \rightarrow \mathbb{R} \) is a deadzone function given by any \( C^1 \) function such that \( q(s) \geq 0 \) for all \( s \) and \( q(s) = 0 \) for all \( |s| \leq \ell \) and \( |s| = \ell \leq |q(s)| \leq |s - \ell| \) for all \( \ell \leq |s| \leq \ell^* \) with \( \ell < \ell^* \). \( a \) and \( \bar{a} \) are positive real numbers, \( n \) is a positive integer and \( \eta \) is an extra control variable whose dynamics are given by

\[
\dot{\eta} = -h(z),
\]

(12)

where \( h(\cdot) \) is a differentiable function, to be chosen, satisfying \( h(s) = 0 \) for all \( s \in \mathbb{R} \) such that \( |s| \leq \ell \).

A few words to motivate the rationale behind this choice are in order. First of all note that, as far as \( \eta \) is concerned, its role is precisely to introduce an integral action in the voltage dynamics whose objective is to asymptotically estimate the unknown term \( \phi_0 \) which, as motivated in the previous section, is a power term taking into account for power losses. In particular, note that, by the fact that \( h(s) = 0 \) if \( |s| \leq \ell \) and by definition of \( \ell^* \), it turns out that \( \eta \) is a constant term as long as \( |v(t)| \) is far from the boundary of the safe region \([V_m, V_M]\). On the other hand, the function \( h(\cdot) \) will be chosen so that the dynamic of \( \eta \) in the case \( |v(t)| \) approaches the boundary of \([V_m, V_M]\), converges to \( \phi_0 \). As far as the presence of the deadzone function in (11) is concerned, note that if \( v(t) \) is far from unsafe values, namely as \( |z| \leq \ell < \ell^* \), then \( q(\tilde{z}) \equiv 0 \) and the power reference signal (11) reduces to \( x^* + \eta \) which, asymptotically, is expected to recover the desired reference (4). On the other hand, if \( |z| \) becomes larger than \( \ell \) approaching the unsafe region then the active power reference signal is enriched with a stabilizing term \( q(\tilde{z}) \) which (as shown in the following analysis) aims to keep the voltage value within the allowed region. In summary the idea is to steer the closed-loop dynamics towards a desired steady state in which \( \tilde{z} \) oscillates within the deadzone (namely within the safe region), and the power \( x \) follows a reference which is \( x^* \) (compensation of the undesired harmonic load components) plus a constant bias (which is precisely the power loss) which is needed in order to make the region \( |z| \leq \ell \) an invariant subspace for the error voltage dynamics. How this can be achieved is the goal of the next part.

Define the error variable for the power dynamics as

\[
\tilde{x} = x - x_{\eta} = x - (x^* - N_1 q(z) + N_1 \eta)
\]

(13)

and for the integral variable \( \eta \) as \( \tilde{\eta} = \eta - \phi_0 \) where \( \phi_0 \) is the smallest solution of (5) (see Section 2.3), so that the overall system (9) and (12), in the new error coordinates, reads as

\[
\dot{\tilde{x}} = M(R, L)\tilde{x} - \frac{1}{L} u + d(t) + I(\tilde{\eta}, \tilde{z}, \tilde{\dot{z}}),
\]

\[
\dot{\tilde{z}} = \nu \tilde{x}^T (\tilde{x} + x^* - N_1 q(z) + N_1 \tilde{\eta} + N_1 \phi_0),
\]

\[
\dot{\tilde{\eta}} = -h(\tilde{z}),
\]

(14)

where

\[
d(t) := d_0 + M(R, L)x^*(t) - \dot{x}^*(t) + M(R, L)N_1 \phi_0
\]

(15)

is a periodic signal with period \( 1/f_s \) and

\[
I(\tilde{\eta}, \tilde{z}, \tilde{\dot{z}}) := MN_1(\tilde{\eta} - q(z) - N_1) \left( \frac{\dot{\tilde{z}} - \frac{d q(z)}{d z} \tilde{z}}{\dot{d}^2} \right).
\]

(16)

Clearly the problem of forcing \( \tilde{x} \) in (14) requires the ability of the control law to compensate for the periodic signal \( d(t) \). Such a compensation can not be achieved in a perfect way by means of a feedforward control of the form \( u = Ld(t) \) as \( L \) and \( d(t) \) are unknown. For this reason we design an internal model-based controller. For, note that each of the two components of the signal \( d(t) \) can be thought as generated by the exosystem

\[
\dot{w}_j(t) = S w_j(t), \quad w_j \in \mathbb{R}^{2N+1},
\]

(16)

\[
d_i(t) = \Gamma_i w_j(t),
\]

where \( \Gamma_i \in \mathbb{R}^{1 \times (2N+1)} \) are suitably defined known vectors and \( S \in \mathbb{R}^{(2N+1) \times (2N+1)} \) is the matrix defined as \( S = \text{blkdiag}(S_j) \) with \( S_0 = 0 \) and

\[
S_j = \begin{pmatrix} 0 & r \omega_j \\ -r \omega_j & 0 \end{pmatrix}, \quad r = 1, \ldots, N,
\]

with the pairs \( (\Gamma_i, S) \) observable. Specifically note that all the uncertainties which characterize \( d_i(t) \) are reflected into uncertainties on the initial state \( w_j(0) \).

With this in mind we define \( \Phi = \text{blkdiag}(S, S) \in \mathbb{R}^{(4N+2) \times (4N+2)} \) and \( \Gamma = \text{blkdiag}(\Gamma_p, \Gamma_q) \in \mathbb{R}^{2 \times (4N+2)} \) and we focus on the internal model-based controller

\[
\tilde{x} = \Phi \tilde{x} + Q \tilde{x}, \quad u = \Gamma \tilde{x} + K \tilde{x},
\]

(17)

where \( Q \) and \( K \) are matrices, of suitable dimensions, to be chosen. We define now the internal model error coordinate as

\[
\xi = \xi - Lw
\]

(18)

where \( w := \text{col}(w_p, w_q) \), so that the overall closed-loop system can be rewritten in the error coordinates as

\[
\dot{\tilde{x}} = M(R, L) \tilde{x} - \frac{1}{L} K \tilde{x} - \frac{1}{L} \Gamma \xi + I(\tilde{\eta}, \tilde{z}, \tilde{\dot{z}}),
\]

\[
\dot{\xi} = \Phi \xi + Q \tilde{x}
\]

(19)
for the \((\tilde{x}, \tilde{z})\) components and
\[ \tilde{z} = \epsilon (Ld^T(t)N_1(\tilde{\eta} - q(\tilde{z})) + \gamma_1(\tilde{z}) + \gamma_2(\tilde{z}) + d_\epsilon(\tilde{z})), \]
\[ \tilde{\eta} = -\epsilon h(\tilde{z}) \] (20)
for the \((\tilde{z}, \tilde{\eta})\) components, where \( \gamma_1(\tilde{x}, \tilde{z}, \tilde{\eta}) := (I \tilde{z} + K \tilde{z})^T(\tilde{x} - N_1 q(\tilde{z}) + N_1 \tilde{\eta}), \gamma_2(\tilde{x}, \tilde{z}, \tilde{\eta}) := (I \tilde{z} + K \tilde{z})^T(\ast(\tilde{z}) + N_1 \tilde{\eta}) + Ld^T(t)\tilde{x} + d_\delta(\tilde{z}) := Ld^T(t)(\ast(\tilde{z}) + N_1 \tilde{\eta}). \)

Note that, by definition of \(d(t)\) in (15) and of \(\phi_0\) as solution of (5), it is easy to see that the forcing the term \(d_\epsilon(\tilde{z})\) is periodic with period \(1/f_\epsilon\) and zero mean value.

As the value of \(\epsilon\) is small, the natural way of approaching the stability analysis of system (19)–(20) is by means of the two-time scale averaging theorem (see Khalil, 1996; Teel, Moreau, & Nesic, 2003). Specifically, we consider the overall system as the interconnection of a fast subsystem represented by the power system with state \((\tilde{x}, \tilde{z})\) and a slow subsystem given by the voltage system with state \((\tilde{\eta}, \tilde{\eta})\) the latter forced, besides by the power variables \((\tilde{x}, \tilde{z})\), by the fast disturbance \(d_\epsilon(\tilde{z})\). In accordance with the general theory of two-time scale averaging systems, we start by studying the boundary layer system, obtained from the overall system by taking \(\epsilon = 0\), which is in our case
\[
\hat{x} = \left( M(R, L) - \frac{1}{L} K \right) \hat{x} - \frac{1}{L} R \hat{z}, \quad \hat{z} = \Phi \hat{x} + Q \hat{x}, \quad \hat{x} = 0, \quad \hat{\eta} = 0. \] (21)

For convenience, denote by \(x_{bl}(t) = (\hat{x}(t), \hat{z}(t))\) and by \(z_{bl}(t) = (\hat{\eta}(t), \hat{\eta}(t))\) the trajectories of (21) with initial conditions \(x_{bl}(0) \in \mathcal{H}_{f,x}, z_{bl}(0) \in \mathcal{H}_{f,z}\), with \(\mathcal{H}_{f,x}, \mathcal{H}_{f,z}\) two compact sets in the respective state spaces. For this system it is easy to prove that there exists a suitable choice of the matrices \(K\) and \(Q\) in (17) such that the subsystem of (21) with state \((\tilde{x}, \tilde{z})\) is Hurwitz. Since the forcing term of this subsystem is constant, this means that \(x_{bl}(t)\) reaches a constant steady state dependent on the initial condition \(x_{bl}(0)\) (i.e., on \(\mathcal{H}_{f,z}\)) and on the DC gain of the Hurwitz system. As far as the latter is concerned, the extra result that it is possible to prove is that the DC gain regarding the \(\hat{x}\) component of \(x_{bl}(t)\) is zero. This is formalized in the next Lemma.

**Lemma 1.** Let \(F_p, F_q \in \mathbb{R}^{(2N+1) \times (2N+1)}\) be two arbitrary Hurwitz matrices and \(G_p, G_q\) two arbitrary column vectors such that \((F_p, G_p)\) and \((F_q, G_q)\) are controllable pairs. Let \(K\) and \(Q\) be defined as
\[
K = k \begin{pmatrix} k_p & 0 \\ 0 & k_q \end{pmatrix}, \quad Q = \begin{pmatrix} E_p^{-1} & 0 \\ 0 & E_q^{-1} \end{pmatrix} \begin{pmatrix} G_p & 0 \\ 0 & G_q \end{pmatrix} K, \]
(22)
where \(k_p, k_q\) are arbitrary positive numbers, \(k\) is a positive design parameter, and \(E_p, E_q\) are the non-singular solutions of the following Sylvester equations:
\[
F_p E_p - E_p S = -G_p \Gamma_p, \quad F_q E_q - E_q S = -G_q \Gamma_q. \]
Let \(\mathcal{H}_{f,x} \subset \mathbb{R}^2 \times \mathbb{R}^{4N+2}\) and \(\mathcal{H}_{f,z} \subset \mathbb{R} \times \mathbb{R}\) be two arbitrary compact sets. Then there exist a vector \(R \in \mathbb{R}^{(4N+2) \times 1}\) and positive \(M_f, \lambda_f, k^*_f\) such that for all \(k \geq k^*_f\) the trajectories of (21) leaving the compact sets \(\mathcal{H}_{f,x}, \mathcal{H}_{f,z}\) satisfy
\[
\| (x_{bl}(t), z_{bl}(t)) \|_{\mathcal{A}_{b1}} \leq M_f e^{-\lambda_f t} \| (x_{bl}(0), z_{bl}(0)) \|_{\mathcal{A}_{b1}},
\]
where
\[
\mathcal{A}_{b1} = \{ (x_{bl}, z_{bl}) \in \mathbb{R}^{4N+4} \times \mathbb{R}^2 : x_{bl} = \begin{pmatrix} 0 & R \end{pmatrix} (\tilde{\eta} - q(\tilde{z})) \}.
\]
**Proof.** Define \(E = \text{blkdiag}(E_p, E_q), G = \text{blkdiag}(G_p, G_q)\) and
\[
R = \begin{pmatrix} -R & 0_{2N} & -\omega_f L \Gamma_q & 0_{2N} \end{pmatrix}^T, \]
where \(\Gamma_p, \Gamma_q\) are, respectively, the first element of \(\Gamma_p\) and \(\Gamma_q\) and \(0_{2N}\) denotes the zero row vector of dimension \(2N\). Consider now the change of variable
\[
\tilde{z} \mapsto \tilde{\chi} := E_{\tilde{z}} - ER_{\tilde{z}} \tilde{\eta} + LG_{\tilde{z}}
\]
which transforms the first two equations of (21) as
\[
\dot{\tilde{x}} = \left( M(R, L) - \frac{1}{L} K + \Gamma E^{-1} G \right) \tilde{x} - \frac{1}{L} \Gamma E^{-1} \tilde{z}, \quad \tilde{\chi} = F \tilde{\gamma} - L(FG - GM(R, L)) \tilde{x},
\]
where \(F = \text{blkdiag}(F_p, F_q)\). From this standard linear arguments can be used to show that the state matrix of the previous system is Hurwitz if \(k\) is large enough from which the claim of the lemma easily follows.

We pass now to study the behavior of the slow subsystem by looking for suitable reduced averaged dynamics, obtained from (20) by confusing the fast dynamics with the steady state of the boundary layer subsystem and the fast disturbance with a suitably defined average, which, since the mean value of \(d_\epsilon(\tilde{z})\) is zero and
\[
f_s L N_1^T \int_{t_{i+1}}^{t_{i+1}+1/f_\epsilon} d(\tau) d\tau = E_{sp} - L \omega_x X_{eq} - R \phi_0
\]
for all \(t \geq 0\), can be easily computed as
\[
\dot{\tilde{z}} = -\epsilon c(q(\tilde{z}) - \tilde{\eta}) - \epsilon R(q(\tilde{z}) - \tilde{\eta})^2, \quad \dot{\tilde{\eta}} = -\epsilon h(\tilde{z}),
\]
(24)
where, by bearing in mind (23),
\[
c := E_{sp} + \phi_0 (R_1^T N_1 - R) = E_{sp} - 2R \phi_0.
\]
(25)
Analyzing the dynamics (24)–(25), it is easily seen that the condition \(c > 0\) is crucial to obtain any stability result. It is simple to verify that \(c\) is indeed strictly positive when condition (7) is verified and the smallest solution of (5) is used to define \(\phi_0\). In particular, according to realistic values of \(R\) and load

\[\text{Here and in the following we denote by } \|x\|_S \text{ the distance of } x \text{ from the set } S \text{ defined as } \|x\|_S = \min_{y \in S} \|x - y\|.\]
power, it is possible to argue the existence of a known positive constant $c^*$ such that $c > c^*$. According to the above considerations, we are able to prove, for the autonomous system (24), that there is a suitable choice of the function $h(\cdot)$ such that the compact set $\mathcal{A}_{dx}$ defined as

$$\mathcal{A}_{dx} := \{(\tilde{z}, \tilde{\eta}) \in \mathbb{R} \times \mathbb{R} : |\tilde{z}| \leq \ell, \tilde{\eta} = 0\}$$

is locally asymptotically stable. This is precisely stated in the next lemma.

**Lemma 2.** Consider system (24) with $h(\cdot)$ chosen as

$$h(s) = r\frac{dq(s)}{ds}q(s),$$

where $r$ is an arbitrary positive number. Let $\mathcal{H}_s(\ell_s) \subset \mathbb{R} \times \mathbb{R}$ be the compact set defined as

$$\mathcal{H}_s(\ell_s) := \{(\tilde{z}, \tilde{\eta}) \in \mathbb{R} \times \mathbb{R} : \|\tilde{z}\|_{\mathcal{A}_{dx}} \leq \ell_s\}$$

and let (7) be satisfied. There exist $\ell_s^* > 0$ and a class $\mathcal{K}$ function $\beta(\cdot, \cdot)$, such that for all $\ell_s \leq \ell_s^*$ the trajectories of (24) with initial conditions in $\mathcal{H}_s(\ell_s)$ satisfy

$$\|(\tilde{z}(t), \tilde{\eta}(t))\|_{\mathcal{A}_{dx}} \leq \beta(\|\tilde{z}(0), \tilde{\eta}(0)\|_{\mathcal{A}_{dx}}, t).$$

**Proof.** Consider the change of variable

$$\tilde{\eta} \mapsto \chi := q(\tilde{z}) - \tilde{\eta}$$

in which $q < 1$ is a positive constant to be chosen which transforms system (24) as

$$\dot{\tilde{z}} = -wc(1-q)q(\tilde{z}) - wc\chi + c hot_1(\tilde{z}, \chi),$$

$$\dot{\chi} = \ell \frac{dq(\tilde{z})}{d\tilde{z}} - c\gamma \chi + q(\tilde{z}) + hot_1(\tilde{z}, \chi)$$

in which $hot_1(\tilde{z}, \chi) = -R((1-q)q(\tilde{z}) + \chi^2)$ is a second order term and $\gamma = r - cg(1-q)$. Now fix $q$ so that $\gamma > 0$ (which is always possible for any $c > 0$). Now consider the following Lyapunov function:

$$V(\tilde{z}, \chi) = \frac{1}{2}q(\tilde{z})^2 + \frac{c}{2}\chi^2$$

which is positive definite with respect to the set $\mathcal{A}_{dx}$. Note that $a\|(q(\tilde{z}), \chi)\|^2/2 \leq V(\tilde{z}, \chi) \leq b\|(q(\tilde{z}), \chi)\|^2/2$, with $a = 1, b = c/\gamma$ if $c/\gamma \geq 0$ or $a = c/\gamma, b = 1$ otherwise. Taking derivative of $V$ along the solutions of (31) yields

$$\dot{V} = \ell \frac{dq(\tilde{z})}{d\tilde{z}} - c\gamma \chi^2 + hot'(q(\tilde{z}), \chi),$$

where $hot'(\cdot)$ is a third order function such that $hot'(0, \chi) = -c\gamma \chi^3/\gamma$. In particular, it turns out that for each $\mu > 0$ there exists $\delta_\mu > 0$ such that

$$\|(q(\tilde{z}), \chi)\| \leq \delta_\mu \Rightarrow hot'(q(\tilde{z}), \chi) \leq \mu \|q(\tilde{z})\|^2 + \mu \|\chi\|^2.$$
In the next part of the section we are interested to have a better insight on the real behavior of the trajectories of system (19), (20). The main goal is to go beyond the result of Proposition 1 by showing how the main control objective of having \( \hat{x} \) approaching zero is indeed satisfied not only in a “practical sense” but indeed in an “asymptotic sense” on finite time intervals of length \( T/\epsilon \) with \( T \) arbitrarily large.

To make this claim more precise, it turns out useful to have a closer look to (20) and to study its trajectories by focusing on the more friendly reduced averaged description (24). This makes sense in the light of the forthcoming result proving that indeed the two trajectories are \( \epsilon \)-close on compact time intervals of length \( T/\epsilon \) with \( T \) arbitrarily large.

Define \( x := \text{col}(\hat{x}, \vec{\tilde{z}}) \), \( z := \text{col}(\vec{\tilde{z}}, \vec{\eta}) \) and rewrite the full order system (19)-(20) in the more compact way

\[
\dot{x} = Hx + B_1\sigma(z) + B_2\epsilon r_1(z, x, t), \quad \dot{z} = \epsilon r_2(z, x, t)
\]  

(34)

in which \(\sigma(z) := \vec{\eta} - q(\vec{\tilde{z}})\), and the functions \(r_1(\cdot, \cdot, \cdot)\), \(r_2(\cdot, \cdot, \cdot)\) and the matrices \(H, B_1, B_2\) are suitably defined. In particular, note that \(H\), according to Lemma 1, is Hurwitz. Consider now the change of variable \(x \mapsto y := x - R\sigma(z)\) with \(R = -H^{-1}B_1\)

transforming (34) into

\[
\dot{y} = Hy + B_2f_1(z, y, t), \quad \dot{z} = \epsilon f_2(z, y, t),
\]  

(35)

where \(f_1(z, y, t) = r_1(z, y + R\sigma(z), t)\) and \(B = B_2 - R\). In this coordinates setting the reduced order system is described by

\[
\dot{z}_{rd} = \epsilon f_2(z_{rd}, 0, t).
\]  

(36)

In turn, this is a \((1/f_i)\)-periodic system whose averaged version is described by (24). In the following we shall denote by \((y(t), z(t)), z_{rd}(t)\) and \(z_{rd}(t)\), respectively, the trajectories of the full system (35), of the reduced system (36) and of the reduced averaged system (24) all originating from the same initial condition \((y(0), z(0))\).

**Lemma 3.** For any \(T > 0\) there exist \(\epsilon^* > 0\) and \(\kappa > 0\) such that for any positive \(\epsilon \leq \epsilon^*\) the following holds:

\[
\|z_{rd}(t) - z(t)\| \leq \kappa \epsilon, \quad \forall t \in [0, T/\epsilon].
\]

**Proof.** Note that standard results about averaging (see, for instance, Khalil, 1996, Theorem 8.3) can be used to prove closeness of solutions, on the time interval \([0, T/\epsilon]\), between the reduced system (36) and the reduced averaged version (24). From this, we will prove the result of the lemma by showing that also the trajectories \(z_{rd}(t)\) are “close,” on arbitrary large compact time intervals of length \(T/\epsilon\), to the trajectories of the full slow subsystem described by the second in (35). To this purpose we follow the steps behind the proof of the well-known Tikhonov theorem (see Khalil, 1996, Theorem 9.1) which, though, cannot be used off-the-shelf in our context as it is not conclusive, as classically stated, about of closeness of solutions on \(\epsilon\)-dependent time intervals. It turns out, however, that the special form of system (35) makes the result true as precisely shown in the following.

Let \(A(t) = z(t) - z_{rd}(t)\) and note that

\[
\|A(t)\| \leq \epsilon \int_0^t \|f_2(z(s), y(s), s) - f_2(z(s), 0, s)\| \, ds + \epsilon \int_0^t \|f_2(z(s), 0, s) - f_2(z_{rd}(s), 0, s)\| \, ds \\
\leq \epsilon T \|f_2\| \sup_{s \leq T} \|y(s)\| + \epsilon T \|f_2\| \sup_{s \leq T} \|z_{rd}(s) - z_{rd}(0)\|
\]

for some positive numbers \(\ell_1\) and \(\ell_2\). Now note that, as a consequence of Proposition 1, there exists a \(\delta > 0\), independent on \(\epsilon\), such that \(\|f_1(z(t), y(t), t)\| \leq \delta\) for all \(t \geq 0\). From this and the fact that \(H\) is Hurwitz, it follows that

\[
\|y(t)\| \leq M e^{-\lambda t} + \epsilon \delta, \quad \forall t \geq 0
\]

for some positive \(\lambda\) and \(M\) which, embedded in the previous estimate of \(\|A(t)\|\), immediately yields

\[
\|A(t)\| \leq \epsilon T \|f_2\| + \epsilon T \|A(t)\| \\
\leq \epsilon T \|f_2\| \sup_{s \leq T} \|y(s)\| + \epsilon T \|A(t)\| \\
\leq \epsilon T \|f_2\| \sup_{s \leq T} \|y(s)\| + \epsilon T \|A(t)\| \\
\leq \epsilon T \|f_2\| \sup_{s \leq T} \|y(s)\| + \epsilon T \|A(t)\| \\
\leq \epsilon T \|f_2\| \sup_{s \leq T} \|y(s)\| + \epsilon T \|A(t)\| \\
\leq \epsilon T \|f_2\| \sup_{s \leq T} \|y(s)\| + \epsilon T \|A(t)\|
\]

which proves the desired result with \(\kappa = (\ell_1 + T) e^{\ell_2 T}\). \(\Box\)

Motivated by this result we focus on the reduced averaged system (24) whose phase portrait is qualitatively sketched in Fig. 2. Looking at this, the trajectories of (24) can be thought as constituted by a number of subsequent trajectory cycles, parameterized in the figure by the parameter \(i\), each of them given by a part with \(|\vec{z}| \leq \ell\) followed by a part confined outside the deadzone region. For the subparts of trajectory within the deadzone region (characterized in the figure by get-in and get-out times, respectively, \(t_1^i, t_2^i\)), (24) simplifies as \(\vec{\eta} = 0\) and \(\vec{z} = -e^{-R\vec{\eta}}\vec{\eta}\) yielding \(\vec{\eta}(t) = \vec{\eta}(t_1^i) = \text{const}\) for all \(t \in [t_1^i, t_2^i]\). Moreover, for all the initial conditions in \(\mathcal{X}_s(\ell_i)\), the arguments in the proof of Lemma 2 can be used to conclude that \(e^{-R\vec{\eta}}(t) > 0\) for all \(t \geq 0\) from which sgn \(\vec{z}(t) = -\text{sgn} \vec{\eta}(t)\) for all \(t \in [t_1^i, t_2^i]\). From this, direct integration of the \(\vec{z}\)
dynamic, yields that the time spent inside the deadzone region at each trajectory cycle is given by
\[ t^1_i - t^1_{i+1} = \frac{2\ell}{e(c - R\tilde{\eta}(t^1_i))\tilde{\eta}(t^1_i)}. \] (37)

On the other hand, in the time intervals \([t^2_i, t^2_{i+1}]\) in which \(|\tilde{z}| > \ell\), the same analysis used in the proof of Lemma 2 shows that the Lyapunov function \(V(t) = V(\tilde{z}(t), \chi(t))\) in (32) (with \(\chi\) defined in (30)) is strictly decreasing and, specifically, \(V(t^2_{i+1}) < V(t^2_i)\). In particular, as \(V(t^2_i) = \tilde{\eta}^2(t^2_i)/2\) and \(V(t^2_{i+1}) = \tilde{\eta}^2(t^2_{i+1})/2\), this implies that \(|\tilde{\eta}(t^2_{i+1})| < |\tilde{\eta}(t^2_i)|\). The decreasing rate of \(\tilde{\eta}\) in the time interval \([t^2_i, t^2_{i+1}]\) can be approximated by computing at the first approximation of system (24) and by considering the simplified case in which \(q(\tilde{z}) = \tilde{z} - t\,\text{sgn} \, \tilde{z}\) for \(|\tilde{z}| > \ell\) (namely a piece-wise linear deadzone function). In this case an elementary calculation yields
\[ |\tilde{\eta}(t^2_{i+1})| \approx -\tilde{\eta}(t^2_i) e^{-\epsilon \ell/2}, \quad t^2_{i+1} - t^2_i \approx \frac{\tilde{\eta}(t^2_i)}{e}. \] (38)

with \(\epsilon = \pi/\sqrt{cE_{sp} - c^2/4}\). As a consequence, by bearing in mind (37), the fact that \(c - R\tilde{\eta}(t) > 0\) for all \(t \geq 0\) and that \(|\tilde{\eta}(t^2_{i+1})| < |\tilde{\eta}(t^2_i)|\), it turns out that the time spent by the reduced averaged trajectory within the deadzone region in two subsequent trajectory cycles is strictly increasing, namely \(t^2_{i+1} - t^2_i > t^2_{i+1} - t^2_i \) and \(\lim_{t \to \infty} (t^2_i - t^2_{i+1}) = \infty\), while the time in which \(|\tilde{z}| \geq \ell\) is independent of \(t\). According to the previous discussion (see (37), (38)), the time period linked to each trajectory cycle is \(T/\epsilon\) in which \(T\) depends, besides on physical parameters, on the value of \(\tilde{\eta}\) but not on \(\epsilon\).

The previous analysis on the reduced averaged trajectories \(\tilde{z}(t)\) allows us also to capture information about the qualitative behavior of the slow dynamics \(\tilde{z}(t)\) of the full order system which, according to Lemma 3, are \(\epsilon\)-close to the behavior of the reduced averaged dynamics. Within each trajectory cycle of the reduced averaged trajectory, it is possible to associate a portion of the slow dynamics of the full system entirely confined in \(|\tilde{z}| < \ell\), for a time interval whose length is increasing as two subsequent cycles are considered, provided that \(\epsilon\) is sufficiently small. More specifically, by bearing in mind the result of Lemma 3, let \(\epsilon\) such that \(\kappa \epsilon \ell/2\) and denote by \(\tilde{z}(t)\) the \(\tilde{\eta}\) and \(\tilde{\eta}\) component at time \(t\) of the reduced averaged system and of the full system, respectively. Then, defining \(\tilde{t}^2_i = \tilde{t}^2_i + n\) and \(\tilde{t}^2_i = \tilde{t}^2_i - n\) with \(n = \kappa/c - R\tilde{\eta}(t^2_i))\tilde{\eta}(t^2_i)\), it turns that for all \(i\) such that \(\tilde{t}^2_i \leq T/\epsilon\) then \(|\tilde{z}(t)| \leq \ell\) for all \(t \in [\tilde{t}^2_i, \tilde{t}^2_{i+1}]\) and, by using (37),
\[ \tilde{t}^2_{i+1} - \tilde{t}^2_i \geq \frac{\ell}{e(c - R\tilde{\eta}(t^2_i))\tilde{\eta}(t^2_i)}. \] (39)

Since the sequence \((c - R\tilde{\eta}(t^2_i))\tilde{\eta}(t^2_i)\) is strictly decreasing as \(i \to \infty\), we conclude that also the slow dynamics of the full system is characterized by subsequent time intervals (39) of increasing length in which the trajectory is entirely confined in the deadzone region. Furthermore, in the same time intervals in which \(|\tilde{z}| \leq \ell\), the fast (current) dynamics (19) of the full order system, described in compact way by the first equation of (34), simplifies as \(\dot{x} = Hx\) with \(H\) Hurwitz. Thus \(x(t)\), and in particular \(\dot{x}(t)\), decay asymptotically to zero with a time constant which is negligible with respect to the length of the interval \(\tilde{t}^2_{i+1} - \tilde{t}^2_i\). In summary the time behavior of the current error \(\tilde{x}(t)\) is characterized by recurrent subsequent cycles constituted by a first constant time interval (only dependent on \(\epsilon\)) in which the \(\tilde{x}(t)\) can be estimated as in the first of (33), and a second time interval (of increasing length for subsequent cycles), in which \(\tilde{x}(t)\) decays asymptotically to zero with arbitrarily fast convergence rate.

**Remark.** Clearly the above analysis holds on the compact time interval \([0, T/\epsilon]\) on which Lemma 3 guarantees closeness of solutions between the full slow system and its reduced averaged version. As the duration of each trajectory cycle is \(T/\epsilon\) (with \(T\) independent of \(\epsilon\)), the number of cycles for which the analysis holds can be rendered arbitrarily large by increasing \(T\) namely, according to Lemma 3, by decreasing \(\epsilon\).

### 4. Experimental results

Experimental tests have been performed to validate the proposed solution (see also Ronchi, Tilli, & Marconi, 2003 for additional details and results). The SAF is characterized by \(L = 3\,\text{mH} \pm 20\%, C = 4400\,\mu\text{F}\) and by an uncertain parasitic resistance of about \(0.12\Omega\). The voltage limits for the DC-link capacitor bank are \(V_m = 700\,\text{V}, V_M = 900\,\text{V}\), while the maximum current amplitude for each inverter leg is \(I_{max} = 100\,\text{A}\). IGBT and free-wheeling diodes have been used for the inverter switches. The parameters characterizing the electric mains are \(V_s = 310\,\text{V}\) (yielding \(E_{sp} = 310^2\,\text{V}^2\)) and \(f_s = 50\,\text{Hz}\).

The load currents, \(i_a, i_b, i_c\), and the filter currents, \(i_a, i_b\) have been measured by means of closed-loop Hall sensors. The line

![Load current, phase a](image)

![Line grid current and voltage, phase a](image)
voltages have been detected using voltage transformers, while the DC-link voltage has been measured with a voltage closed-loop Hall sensor. A digital control board based on the DSP TMS320C32 has been used to implement the proposed control strategy. The adopted sampling frequency $f_{\text{samp}} = 1/T_{\text{samp}}$ is equal to 7 kHz. The proposed nonlinear continuous-time solution has been discretized according to the procedure described in Ronchi et al. (2003) for the power controller, while Euler method has been adopted for the DC-link voltage stabilizer. A PWM technique with a carrier frequency equal to $f_{\text{samp}}$ has been used for implementation.

A balanced nonlinear load is the benchmark for the proposed solution. It is the sum of an electric induction motor with no load and of a lamp bank fed by means of a diode rectifier. Its $a$-phase current is shown in the top picture of Fig. 3. The relevant load power harmonics are the 6th and the 12th.

The control parameters of the internal-model-based power controller have been selected as follows $S = \text{blkdiag}(S_0, S_6, S_{12}), \Gamma_p = \Gamma_q = (1, 1, 0, 1, 0)^T, K = \text{diag}(200, 200, 200)$ and $Q = 10^3 \cdot \text{blkdiag}(Q_p, Q_q)$ with $Q_p = Q_q = (40.6, 80.7, 7.15, 78.7, 17.6)^T$. The internal model has been selected according to the relevant load harmonics to reduce the computational burden. The voltage stabilizer has been implemented using $\ell = 0.8(V_M^2 - \bar{V}^2), q(s) = 0\forall|s| \leq \ell; q(s) = \text{sgn}(s)|s - \ell|^2\forall|s| > \ell$ and $r = E_{sp}/8$.

The $a$-phase current drawn from the line after SAF compensation is reported in the bottom picture of Fig. 3 in which the residual ripple is mainly related to the PWM effects. The fast Fourier transform of the $p-q$ components of load power and power drawn from the main after SAF compensation is shown in Fig. 4. It can be noted that the constant component of the

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**Fig. 4.** FFTs of the $p$ and $q$ components of load power (dark bar) and power drawn from the main after SAF compensation (white bar) expressed in [W].

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**Fig. 5.** First and second plot: Error variables ($\tilde{i}$ [W], $\tilde{z}$[V$^2$]) of the SAF controller during experiments; third plot: command generated by the DC-link voltage controller [V].
load imaginary power and the 6th and 12th load current harmonics are fully eliminated, as expected, whereas the 18th is slightly amplified. The reason of this phenomenon is related to the digital implementation of the power controller as discussed in Ronchi et al. (2003). This behavior can be completely avoided with a suitable model-based pre-filtering of the power references related to the load harmonics (see Tilli et al., 2002). The constant component of the real power \( x_p \) is slightly larger at line side, due to the additional active power drawn to compensate for SAF losses and stabilize the DC-link voltage dynamics. In Fig. 5 the dynamic behavior of the controlled SAF is reported. From the first and third pictures, it can be noted as \( \tilde{x} \) is negligible even when the voltage control actions \( q \) (continuous line) and \( \eta \) (dashed line) are varying. This qualitatively confirms the time-scale separation between \( (\tilde{x}, \tilde{z}) \) and \( (\tilde{q}, \tilde{\eta}) \). Obviously \( \tilde{x} \) is not exactly zero, even when \( q(\tilde{z}) \) and \( \eta \) are constant, due to the high-order load harmonics not compensated by the internal model (e.g. the 18th) and the measurement noise. From second picture it can be noted that \( v^2 \) bounces between the deadzone limits \( \bar{V} - \ell, \bar{V} + \ell \), according to the discussion presented in second part of Section 3. Inside the deadzone the \( \tilde{z} \) is basically driven by \( \tilde{\eta} \) (according to averaging results). At each impact with the deadzone bounds \( \tilde{\eta} \) decreases yielding that the time interval between two subsequent hits of the bounds increases.

The oscillations of the DC-link voltage at frequency \( f_i \) are not clearly visible in Fig. 5, since the load harmonics of the benchmark load are very small with respect to the ratings of the adopted SAF. Fig. 6 shows a zoom of \( x, x^{\ast} \) and \( z = v^2 \) to highlight the structural oscillations on \( z \) and the behavior of power variables. The offset between \( x_p \) and \( x^{\ast}_p \) is the command generated by the DC-link voltage controller.

5. Conclusions

The design of an internal model-based nonlinear robust control of a shunt active filter has been proposed to compensate for the load harmonics and imaginary power component and to stabilize voltage internal dynamics. The stability proofs rely upon a two-time scale assumption which is naturally enforced by the physical parameters of the system. Experimental results have been presented to validate the proposed approach.

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References


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